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"OPTICAL TURBULENCE" IN LASER RADIATION PROBLEMS

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A new version of describing the process of excitation of quasistochastic self-oscillations in a system of the "reaction–diffusion" type with nonlinear boundary conditions is presented.

In this work it is shown that the properties of boundary surfaces exert a substantial influence on the radiation-density distributions and in some cases lead to the appearance of "optical turbulence" [1]. We present below a modified version of the mathematical results reported in [2-4] and their application to boundary-value problems of nonlinear optics [5].

Formulation of the Problem. For radiation transfer equations, the following parabolic approximation can be obtained [6, 7]:

$$\frac{1}{v_1}\frac{\partial J_{1\lambda}}{\partial t}+\frac{\partial J_{1\lambda}}{\partial x}=\chi_1\frac{\partial^2 J_{1\lambda}}{\partial x^2}+F_1(J_{1\lambda}),\quad \frac{1}{v_2}\frac{\partial J_{2\lambda}}{\partial t}-\frac{\partial J_{2\lambda}}{\partial x}=\chi_2\frac{\partial^2 J_{2\lambda}}{\partial x^2}+F_2(J_{2\lambda}).\tag{1}$$

To solve system (1), we consider the classical initial-boundary-value problem in the domain $\Pi=\{x,t\colon 0\le x\le 1,\ t\ge 0\}$ with the boundary conditions

$$\frac{\partial J_{i\lambda}}{\partial x}=\alpha_i(J_{1\lambda},J_{2\lambda})\big|_{x=0},\quad \frac{\partial J_{i\lambda}}{\partial x}=\beta_i(J_{1\lambda},J_{2\lambda})\big|_{x=1},$$

where α_i and β_i are prescribed nonlinear functions, $i=1,2$, and with certain initial conditions:

$$J_i(x,0)=h_i(x),\quad 0<x<1.$$

It is known ([6], p. 290) that in a one-component active medium, for instance, for the Kolmogorov–Petrovskii–Piskunov equation

$$\tau\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}+f(u)$$

the only possible type of autowave processes is switching waves that occur in a nonlinear cavity; at the same time, for multicomponent active media, various autowave processes can take place, e.g., Turing structures, master centers [7], dynamic chaos, and so on.

Below we restrict ourselves to an investigation of the solutions that have the form of running waves with arguments of the form $(x-(\omega/k)t)$. Such processes can be said to have the phase velocity $v=\omega/k$.

Reduction to a Difference Equation. Stationary solutions of system (1) have the form

$$\tilde u(x,t)=u(\xi,t);\quad \tilde v(x,t)=v(\eta,t);\quad c=1,$$

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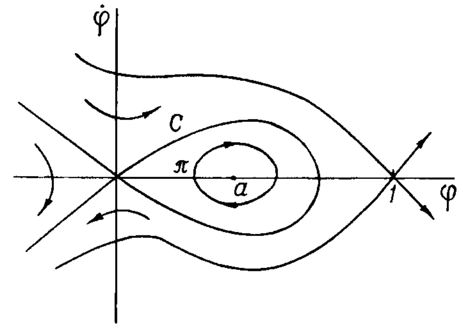


Fig. 1. Phase picture of a finite-dimensional oscillator; a, center-type point; C, separatrix dividing periodic and "drift" trajectories.

where $\xi = t - x$ and $\eta = t + x$. Then Eqs. (1) are reduced to the system of ordinary differential equations

$$\chi_1 u''(t+x) + F_1(u(t+x)) = 0, \quad \chi_2 u''(t-x) + F_2(u(t-x)) = 0. \tag{2}$$

Note that any solution $(u, v) = (\varphi_1, \varphi_2)$ satisfies the equations

$$\frac{1}{2}\dot{\varphi}_i^2(s_i) + \Psi_i(\varphi_i(s_i)) = c_i,$$

where

$$\Psi_i(\varphi_i) = \int_0^{\varphi_i} F_i(\tilde{\varphi}_i)\, d\tilde{\varphi}_i;$$

$s_i = (t + (-1)^{i-1}x)$; c_i are some constants; thus, each c_i corresponds to its "own" phase space (φ_i, φ_i'); here, one of the the level lines depicted in Fig. 1 will be an orbit, $i = 1, 2$.

We determine

$$\varphi_1(s_1) = h_1(s_1), \quad s_1 = x, \quad 0 \le x \le 1;$$

$$\varphi_2(s_2) = h_2(s_2), \quad s_2 = -x, \quad -1 \le s_2 < 0,$$

so that the family of trajectories $\gamma(\cdot)$ that depend on x as a parameter will correspond to the set of initial functions $(h_1, h_2) \subset I_1 \times I_2$, where I_1 and I_2 are some open bounded intervals: for instance, the family of periodic solutions will correspond to the set $\pi \subset I_1 \times I_2$ inside the region bounded by the curve C [8]; more complicated situations are also possible [9].

Note that for the set $\gamma_0(h_1) \subset \gamma(c_1)$ we obtain a single periodic trajectory (Fig. 1); hereafter it will be assumed that this requirement is always fulfilled and we call it condition (α). Then the level line can be determined uniquely:

$$c_i = \dot{h}_i^2(0) + \Psi_i(h_i(0)), \quad i = 1, 2,$$

and by virtue of the aforesaid the following relations stem from Eqs. (1):

$$\dot{\varphi}_1^2(t-1) + \Psi_1(\varphi_1(t-1)) = c_1, \quad \dot{\varphi}_2^2(t+1) + \Psi_2(\varphi_2(t+1)) = c_2 \tag{3}$$

and

$$\dot{\varphi}_1^2(t) + \Psi_1(\varphi_1(t)) = c_1, \quad \dot{\varphi}_2^2(t) + \Psi_2(\varphi_2(t)) = c_2. \tag{4}$$

Hence the existence of 1-periodic solutions in the variable t follows immediately for the Cauchy problem:

$$\varphi_i(t) = \varphi_i(t-1)\,,\quad i = 1, 2\,;\quad t \in [-1, \infty)\,.$$

Next, the equalities

$$\frac{\partial \varphi_1}{\partial \xi} = \frac{\partial \varphi_1}{\partial x}\,,\quad \frac{\partial \varphi_2}{\partial \eta} = \frac{\partial \varphi_2}{\partial x} \tag{5}$$

are obvious, and therefore from (1)-(5) and the boundary conditions we can obtain the system of four functional equations

$$\alpha_1^2(\varphi_1(t-1), \varphi_2(t+1)) + \Psi_1(\varphi_1(t-1)) = c_1(1)\,,$$

$$\alpha_2^2(\varphi_1(t-1), \varphi_2(t+1)) + \Psi_2(\varphi_2(t+1)) = c_2(1)\,,$$

and

$$\beta_1^2(\varphi_1(t), \varphi_2(t)) + \Psi_1(\varphi_1(t)) = c_1(0)\,,\quad \beta_2^2(\varphi_1(t), \varphi_2(t)) + \Psi_2(\varphi_2(t)) = c_2(0)\,.$$

Note that for $c_1(1) = c_1(0)$ it follows immediately that (after the shift $t \to t+1$ in the third equation and its subsequent comparison with the first equation) the following equality holds: $\alpha_1 = \beta_1$; if similarly $c_2(1) = c_2(0)$, then $\alpha_2 = \beta_2$ and we arrive at the system of equations

$$\begin{aligned} \alpha_1^2(\varphi_1(t-1), \varphi_2(t+1)) &= c_1 - \Psi_1(\varphi_1(t-1))\,, \\ \alpha_1^2(\varphi_1(t-1), \varphi_2(t+1)) &= c_2 - \Psi_2(\varphi_2(t+1))\,. \end{aligned} \tag{6}$$

Here, for instance, the following situations can take place: 1) for $c_1 = c_2$, $\alpha_1 = \alpha_2$, and $\psi_1 = \psi_2$ we have a family of solutions (obviously, periodic ones) for the level line (Fig. 1); 2) the first of the equations of system (6) is solvable (uniquely or nonuniquely) for any φ_1, $\varphi_2 \in I$ in such a way that

$$\varphi_1 = \gamma_{1,c_1}(\varphi_2)\,,$$

and the second equation is solvable in such a way that

$$\varphi_2 = \gamma_{2,c_2}(\varphi_2)\,,\quad \varphi_1, \varphi_2 \in I\,,$$

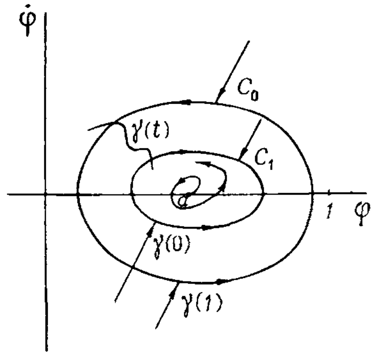
i.e., we obtain a family of constant solutions. If there are several such solutions (Fig. 2): for instance, both φ_1^1 and φ_2^2 are unstable and φ_1^0 is stable, then we have an example of classical switching waves [6]; thus, the situation $\Psi_1 = \varphi_1 - \varphi_1^3$ considered in [7], p. 65 coincides with that described above for

$$\Psi_1(\varphi_1) = \gamma_{1,c_1}(\gamma_{2,c_2}(\varphi_1))\,.$$

Thus, condition (α) implies the equality $c_1(0) = c_1(1)$. Note that this is the rather rigid requirement on the choice of the initial functions

$$\dot{h}_1^2(0) + \Psi_1(h_1(0)) = \dot{h}_1^2(1) + \Psi_1(h_1(1))$$

at the corner points, and henceforward, to obtain more informative results, we will have to reject it (the condition) and deal with a family of level lines $\gamma_1(h_1)$ and $\gamma_2(h_2)$, respectively, so that the set $\gamma_0(h_1(x))$ for $0 \le x \le 1$ generates a "tube of levels" (Fig. 2); here the lines C_0 and C_1 are boundary ones (here, we assume



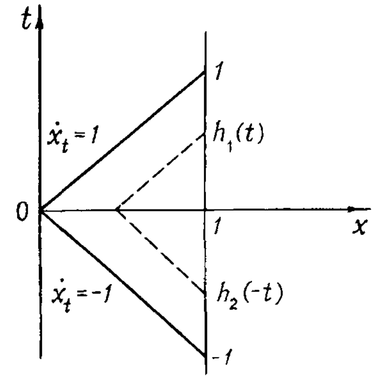


Fig. 2. Family or "tube" of levels (a change in the "parameter" $0 < x < 1$ entails a change in the trajectories in the finite-dimensional phase space; all values of this parameter generate a family or "tube" of levels).

Fig. 3. Illustration of the geometric application of the method of characteristics.

that the boundary conditions are fulfilled), and the phase space ($\dot{u}$, $\dot{v}$, u, v) is the product (Fig. 2) of two disks in R^2, on which, as will be shown below, we can construct a nonlinear representation of the shift along solutions of the initial boundary-value problem that is generated by nonlinear boundary conditions.

Next, we assume that: 3) for any $(\varphi_1, \varphi_2) \in I$, where I is some open bounded interval such that $(h_1, h_2) \in I$, the system of equations (4) is solvable (uniquely or nonuniquely) in such a way that

$$\varphi_1(t+1) = \Phi_1(\varphi_2(t)), \tag{7}$$

and system (3) is solvable in such a way that

$$\varphi_2(t+1) = \Phi_2(\varphi_1(t-1)), \tag{8}$$

where Φ_1 and Φ_2 are some functionals.

We perform the argument shift $t \to t+1$ in functional equation (7) and use the value of φ_2 from Eq. (8):

$$\varphi_1(t+1) = \Phi_1(\Phi_2(\varphi_1(t-1))). \tag{9}$$

For solution of difference equation (9) with continuous time, we prescribe the following initial conditions [10]:

$$h(t) = \begin{cases} h_1(t) & \text{at } 0 \le t \le 1, \\ -h_2(t) & \text{at } -1 \le t < 0, \end{cases} \tag{10}$$

which are obtained by extending the initial functions h_1 and h_2 along the characteristics $\dot{x}_t = 1$ and $\dot{x}_t = -1$, respectively, to the boundary of the region $\{x = 1, t > 0\}$ with subsequent extension of the values of the running-wave amplitudes into the domain of definition of the solutions (Fig. 3).

Thus, the initial boundary-value problem for stationary solutions of the running-wave type is reduced to difference equation (9) with initial conditions (10). The reduction method, from which, in particular, relation (10) stems, is considered in detail in [5, 10]; there, conditions on the functions F_i and Ψ_i and on the initial data h_i, $i = 1, 2$, are given and a theorem on the asymptotic behavior of the solutions of problem (9)-(10) and hence the solutions of the initial boundary-value problem is formulated.

For illustration, it is convenient to consider the classical boundary-value problem with the zero Neuman boundary conditions and functions of the form

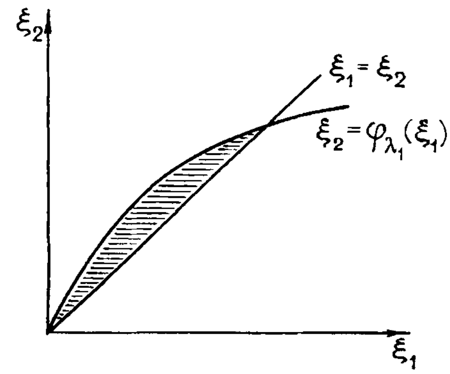


Fig. 4. Domain of existence of bounded solutions.

$$f_1(\xi_1, \xi_2) = \xi_2 - \mu\xi_1(1 - \xi_1) \quad \text{and} \quad f_2(\xi_1, \xi_2) = \xi_1 - \xi_2 .$$

Let us introduce characteristic time intervals and dimensionless variables. Then for the first equation we have

$$\tilde{v} = \frac{\tau_0 v}{L} \quad \text{and} \quad D = \frac{\tau_0 \kappa_1 \tilde{v}}{L^2},$$

and likewise for the second equation. From [10] it follows that the limiting solution is a $2^N/\tilde{v}$-periodic function, where N is the least common multiple of the periods of the attracting cycles of the mapping [10]

$$\varphi_{\tilde{\mu}}\colon\ \xi_1 \to \tilde{\mu}\,\xi_1(1 - \xi_1),$$

which is generated by superposition of the functions f_1 and f_2.

Finally, taking account of diffusion leads to the representation [2]

$$u_1 \to \tilde{\mu}\frac{D_1}{D_2}u_1(1 - u_1), \quad D_i = \kappa_i,$$

where $\varepsilon\alpha = D_1/D_2$, here $\varepsilon = \tau_1/\tau_2$ is a parameter determining the ratio of the characteristic time intervals and $\alpha = \kappa_1/\kappa_2$ is a parameter determining the ratio of the spatial scales ([6], p. 67).

Without restricting the generality, we set $\tilde{\mu} = 1$ and $\kappa = D_1/D_2$. Then from [10] it follows that for $0 < \kappa < 4$ the function φ_κ: $u \to \kappa u(1 - u)$ maps the interval $I = [0, 1]$ into itself. For $0 < \kappa \le 1$ we obtain $\Omega_\kappa^+ = 0$, and for $h_1 = h_2 = 0$ we have $u_1 = u_2 = 0$; here [2] $\Omega_\kappa^+ = \{\xi_1, \xi_2 \ge 0;\ f_i \le 0\}$ is the condition on the active medium of the cavity.

For $1 < \kappa < 3$ the solution component u_1 monotonically approaches β_1, while for $2 \le \kappa < 3$ it oscillates relative to β_1 (obviously, the same can be said of the solution component u_2).

Note that, here, the usual requirement of the existence of the small parameter $D_1/D_2 << 1$ (see, e.g., [7]) for determination of self-oscillating regimes in the vicinity of null isoclines leads to oscillating regimes only for a "supernonlinear" active medium of the cavity, so that $1 < \tilde{\mu}D_1/D_2 \le 4$; the nonlinearity parameter is $\tilde{\mu} > D_2/D_1 >> 1$.

It is also pertinent to note that from the inclusion $h_1, h_2 \in \Omega_\kappa^+$ the inclusion $u_1, u_2 \in \Omega_\kappa^+$ follows. This is a rather "fine" requirement on the choice of the initial functions (Fig. 4).

For $3 \le \kappa < 1 + \sqrt{6}$ the region Ω_κ^+ increases, and each solution with initial functions from Ω_κ^+ is a solution of the relaxation type that is asymptotic relative to some 4-periodic function.

With increase in κ to $\kappa^* = 3.568$ ... the periods of the attracting cycles undergo successive doubling. For $\kappa > \kappa^*$ the limiting solution has an uncountable, nowhere dense set of multivaluedness points: in the terminology of [10] these solutions can be both turbulent and strongly turbulent correspondingly.

The above scenario is in qualitative agreement with existing types of self-similar solutions for systems with optical feedback ([6], p. 316). Such structures are often called nonlinear modes and only in this sense can we speak of "optical turbulence," since the actual turbulence must contain an infinite number of modes, and each self-similar solution has its own region of attraction of the initial data.

The instabilities considered that arise, for instance, when the amplitude of the incident radiation (or the parameter of the laser pumping) changes and lead to a sequence of bifurcations of the corresponding limiting regimes represent, in essence, known Ikeda instabilities (see, e.g., [6], p. 254), where the initial stationary regime becomes nonstationary and generates a regime with period 2, next with periods 4, ..., 2^N, ..., and so on, and at some $\mu > \mu^*$ an autostochastic regime develops. The corresponding physical models can be found, for instance, in [6, 7].

For laser-radiation models, some estimates of the number of photons in a nonlinear mode are made in [11]; thus, for instance, for a typical 2^N-periodic regime of $\tau_0 c/L = 1$ for $\tau_0 = 10^{-10}$ sec, which corresponds to an Na laser [11], we obtain $L = 10^{-4}$ m. Then for comparatively short cavities with plane mirrors, where the Fresnel number is

$$\mathrm{Fr} = a_0^2/\lambda L >> 1 ,$$

where a_0 is the transverse dimension of the light beam at $\lambda = \omega^{-1}$ (here $\omega = 10^{15}$ sec^{-1} for an Na laser) we obtain the rough estimate

$$\mathrm{Fr} = \frac{a_0^2 \omega}{L} = 10^{19}\, a_0^2 >> 1 ,$$

which shows that in this case, where the systems length L is much smaller than the characteristic length of diffraction smearing of the given beam aperture a_0 (Fr >> 1), we can restrict ourselves to the approximation of geometrical optics.

NOTATION

v_i, phase velocities; $J_{i\lambda}$, components of the radiation densities; D, diffusion coefficient in the Kolmogorov–Petrovskii–Piskunov (KPP) equation; ω, frequency; k, wave number; λ, wavelength; κ_i, diffusion coefficients in the radiation transfer equations; i_λ, index of the monochromatic-radiation wavelength; i, index of the number of components of the solution; α_i, β_i, functions reflecting the nonlinear interaction of the optical-medium boundary with the external medium; h, initial function; f, nonlinear source in the KPP equation; $\gamma(\cdot)$, curve in the phase space; $'$, derivative; $\cdot$, derivative; c_i, some constants; $\gamma_{1,2}$, analytical representation of functions and/or their superposition, some curves in the geometric context; Φ_i, functions in the difference equation; ξ_i, arguments of the functions considered in the example; $\tilde{\mu}$, parameter of the function in the example; κ, modified physical parameter of the same function; *, index of the parameter at whose value quasistochastic oscillations are possible; a_0, transverse dimension of the light beam.

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